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A PROBLEM OF APPROACH WITH TWO DIFFERENT PURSUERS AND ONE EVADER*

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The problem of the simple pursuit of an object in a plane by two other objects is considered. It is assumed that the pursuers' maximum velocities satisfy different bounds, while the evader moves at most as rapidly as the slower pursuer. The duration of the game is fixed. The payoff functional is the distance between the evader and the nearest pursuer at the end of the game. Without using the explicit form of the programmed maximin function, it is proved that the function is u-stable throughout the space, i.e., it is identical with the value of the differential game. It is also proved that the programmed absorption time equals the optimum response time.

In a previous paper /1/, an optimum solution was found for this approach-game problem when the pursuers' velocities are bounded by the same quantity.

This paper continues the investigations of /1-3/. The optimum response problem was solved in /4/ for the case of identical pursuers. The formalization of differential games employed here is that of /5-7/.

1. Suppose the motions of the fast pursuer $S(x)$, the slow pursuer $Q(y)$ and the evader $E(z)$ are described by the equations

$$x_i^{\cdot} = u_i, \quad y_i^{\cdot} = v_i, \quad z_i^{\cdot} = w_i; \quad i = 1, 2 \tag{1.1}$$

The pursuers' and evader's control vectors satisfy the bounds

$$(u_1^2 + u_2^2)^{1/2} \leqslant \mu, \quad (v_1^2 + v_2^2)^{1/2} \leqslant \lambda, \quad (w_1^2 + w_2^2)^{1/2} \leqslant \nu \tag{1.2}$$

the pursuers' superiority being guaranteed by the condition

$$\nu \leqslant \mu < \lambda \tag{1.3}$$

The game is considered in the time interval $[t_0, \vartheta]$. The payoff functional is the Euclidian distance between the evader and the nearest pursuer at the time the game ends, $t = \vartheta$, i.e.,

$$\sigma = \min\{\|z(\vartheta) - x(\vartheta)\|, \ \|z(\vartheta) - y(\vartheta)\|\} \tag{1.4}$$

The phase space of the system is in fact three-dimensional. For a full description of the game position at any given time, it suffices to know the triple of numbers (x, y, z), where $z = \|SQ\|$ and (x, y) are the Cartesian coordinates of E in a coordinate frame attached to the pursuers (Fig.1). The degenerate case $S(t_0) = Q(t_0)$ reduces to a game of approach between the fast pursuer Q and the evader E, whose value is known (we denote it by ρ^{11}). In the sequel it will be assumed that $S(t_0) \neq Q(t_0)$.

The dynamics of the phase vector of relative coordinates $\zeta = (x, y, z)$ are described by the following system of differential equations:

$$\begin{aligned} x^{\cdot} &= w_1 - (u_1 + v_1)/2 + y(v_2 - u_2)/z \\ y^{\cdot} &= w_2 - (u_2 + v_2)/2 - x(v_2 - u_2)/z \\ z^{\cdot} &= v_1 - u_1 \end{aligned} \tag{1.5}$$

The players' domains of attainability are represented in the game plane by circles G_s, G_q, G_e of radii $R_s = \mu(\vartheta - t_0)$, $R_q = \lambda(\vartheta - t_0)$, $R_e = \nu(\vartheta - t_0)$, centred at the points S, Q and E, respectively. Let m be the locus of all points equidistant from the circles G_s and G_q. Denote the points at which the circle G_e cuts m by A^* and A_* (Fig.2). In the cases $G_e \cap m = \{\varnothing\}$ or $A^* = A_*$ it can be shown that the original game degenerates into a two-person game with value ρ^{11} equal to the programmed maximin γ.

Let $A^* \neq A_*$. If $E \notin \operatorname{int} SA^*QA_*$, the game again reduces to a two-person game (the domain in which $\rho^{21} = \rho^{11}$ is denoted by D_{11}). The domain in which $E \in \operatorname{int} SA^*QA_*$ is denoted by D_{21}. Anticipating, we remark that in the domain D_{21} the pursuers can increase

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their chances of success by interacting.

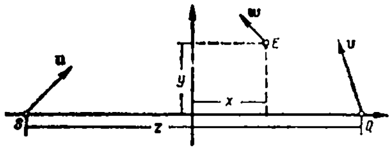

Fig.1

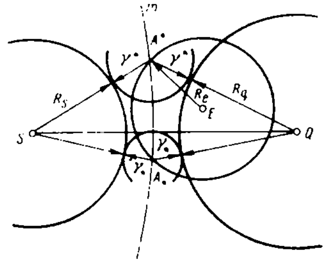

Fig.2

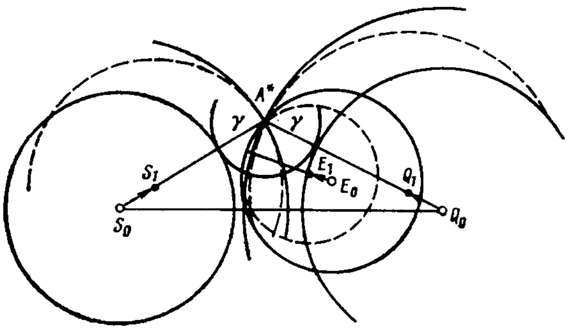

Fig.3

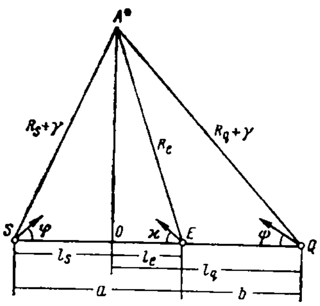

Fig.4

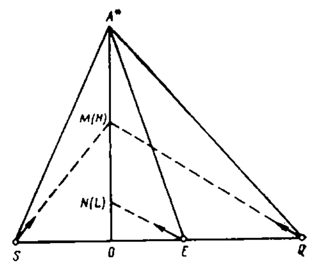

Fig.5

The programmed maximin function γ is defined as follows:

a) $\gamma = 0$ if $G_s \cup G_q \supset G_e$;

b) if $\gamma^* = \rho(A^*, G_s) = \rho(A^*, G_q)$ and $\gamma_* = \rho(A_*, G_s) = \rho(A_*, G_q)$, then

$$\gamma = \max\{\gamma^*, \gamma_*\} \text{ if } G_e \setminus (G_s \cup G_q) \neq \{\varnothing\}$$

($\rho(\cdot,\cdot)$ is the Euclidian distance from a point to a set in the plane). Note that a system of equations can be set up for the functions γ^* and γ_*, solving the triangles SA^*Q, SA_*Q and SEQ, but the cumbersome form of the resulting expressions hinders their effective investigation.

It can be seen from the definition that the function γ is piecewise-smooth in D_{31}, it may fail to be smooth only on the surface $y = 0$ (when $\gamma = \gamma^* = \gamma_*$). It will be shown later that the programmed maximin function γ is u-stable throughout space and is therefore identical with the value of the game /6/.

2. Let $\gamma(t_0, \zeta_0)$ be the programmed maximin at a position $\{t_0, \zeta_0\} \in D_{31}$; to fix ideas, suppose that $E_0 \in \Delta S_0 A_0^* Q_0$ $(y_0 > 0)$. Define an extremal motion of players S and Q as a motion with maximum velocity at the point A^*. Denote the corresponding control constants of the players by u° and v°. Then there exists a number T such that, for any $\Delta t < T$ and an arbitrary measurable sample $w(t)$ the control triple $(u^\circ, v^\circ, w(t))$ takes the system at time $t_1 = t_0 + \Delta t$ to position $\{t_1, \zeta_1\}$, in such a way that, for every $t \in [t_0, t_1]$, $E(t) \in \Delta S(t) A^* Q(t)$. At the initial position (at time $t = t_0$) we have a covering

$$G_s(t_0) \oplus \gamma C \cup G_q(t_0) \oplus \gamma C \supset G_e(t_0)$$

which obviously remains valid at time $t = t_1$ as well (Fig.3) (C denotes the unit circle about the origin). This means that $\gamma(t_1, \zeta_1) \leqslant \gamma(t_0, \zeta_0)$, which is precisely the condition for u-stability.

3. We will now investigate the behaviour of the function γ on the surface $y = 0$. Let us assume that either $\gamma > 0$, or $\gamma = 0$ but $A^*, A_* \in \partial(G_s \cup G_q)$. The stability of the function $\gamma = 0$ when $A^*, A_* \in \operatorname{int}(G_s \cup G_q)$ is obvious.

It is convenient to conduct our investigation of γ on the surface $y = 0$ in a special coordinate frame (Fig.4). Let $a = \| SE \|$, $b = \| EQ \|$, $c = a + b$. Relative to this frame, the value of γ is determined from the formula

$$\gamma = (R_e^2 + ab(1 - ((R_s - R_q)/c)^2))^{1/2} - (R_s b + R_q a)/c \tag{3.1}$$

As system (1.5) is linear in the controls, it will suffice to verify the u-stability of γ for $\| w(t) \| = \nu$. Moreover, stability will be established if, for any choice of the control $w(t)$ $(\| w(t) \| = \nu)$ by player E at time t, there exist $\delta > 0$ and controls $u(\tau)$, $v(\tau)$ of the pursuers in the interval $\tau \in [t, t + \delta]$ (with $\| u \| = \mu$, $\| v \| = \lambda$) such that the condition $d\gamma/dt \leqslant 0$ holds along the corresponding trajectory of motion of the system.

Suppose that player E's position is to the right of the perpendicular A^*O $(a > \| SO \|)$. Introduce angles as new controls (Fig.4):

$$u_1 = \mu \cos\varphi, \quad v_1 = -\lambda\cos\psi, \quad w_1 = -\nu\cos\varkappa$$

$$u_2 = \mu\sin\varphi, \quad v_2 = \lambda\sin\psi, \quad w_2 = \nu\sin\varkappa$$

(if $a \leqslant \| SO \|$ we replace $\varkappa \mapsto -\varkappa$). Consider the equations of motion in coordinates (a, b, y):

$$\begin{aligned} a^{\cdot} &= -(\mu\cos\varphi + \nu\cos\varkappa), \quad b^{\cdot} = -(\lambda\cos\psi - \nu\cos\varkappa) \\ y^{\cdot} &= \nu\sin\varkappa - b\mu\sin\varphi/c - a\lambda\sin\psi/c \end{aligned} \tag{3.2}$$

Note that the angles corresponding to controls u° and v° in the new notation are φ° and ψ°.

Let $0 \leqslant \varkappa \leqslant \pi$ (for $-\pi \leqslant \varkappa \leqslant 0$ the reasoning is analogous). Divide the remaining part of the vectogram of player E into two subsets:

1) controls $\varkappa(t)$ which, together with φ° and ψ°, generate a motion with $y^{\cdot} > 0$; these controls consitute the set K_1;

2) all other controls - the set K_2.

For controls in the first subset, u-stability of γ can be proved by arguments similar to those of Sect.2.

Consider the second subset of controls for player E. In this case, a trajectory generated by controls $(\varphi^\circ, \psi^\circ, \varkappa)$ is characterized at time $t + \delta$ $(\delta > 0)$ by $y^{\cdot} < 0$ and the point E lies in the lower triangle SA_*Q. We now impose additional constraints on the pursuers' controls. Given a position $(x, 0, z)$, we consider only controls $\varphi(t)$ and $\psi(t)$ which, together with $\varkappa(t) \in K_2$, generate trajectories that slide $(y^{\cdot} = 0)$ along the surface $y = 0$. We shall show that among the remaining controls there are some that generate trajectories along which γ is a non-increasing function.

Thus, the dynamics of the system are described by the first two equations of (3.2). Sliding of the phase trajectory along the surface $y = 0$ is guaranteed by the final condition

$$c\nu\sin\varkappa = b\mu\sin\varphi + a\lambda\sin\psi \tag{3.3}$$

Let $\Phi(\varkappa)$ denote the set of all pairs of controls (φ, ψ) such that $0 \leqslant \varphi \leqslant \varphi^\circ, 0 \leqslant \psi \leqslant \psi^\circ$ and the triple $(\varphi, \psi, \varkappa)$ satisfies condition (3.3).

The u-stability of γ on the surface $y = 0$ follows from the inequality

$$\max_{\varkappa \in K_2} \min_{(\varphi, \psi) \in \Phi(\varkappa)} (d\gamma/dt) \leqslant 0 \tag{3.4}$$

To prove inequality (3.4), differentiate (3.1) along trajectories of system (3.2), observing the constraint (3.3). This gives an expression for the derivative of γ $(\Gamma = d\gamma/dt|_{y^{\cdot}=0})$, with the angles $\varphi, \psi, \varkappa$ satisfying (3.3). Evaluate the minimum of this expression as a function of $(\varphi, \psi) \in \Phi(\varkappa)$:

$$\begin{aligned} &\min\Gamma = \min[c\nu l_e\cos\varkappa - b\mu l_s\cos\varphi - a\lambda l_q\cos\psi] \\ &l_e = R_e\cos\varkappa^\circ, \quad l_s = (R_s + \gamma)\cos\varphi^\circ, \quad l_q = (R_q + \gamma)\cos\psi^\circ \end{aligned}$$

For fixed $\varkappa \in K_2$, the minimum is reached at controls such that

$$l_s \operatorname{tg}\varphi = l_q \operatorname{tg}\psi \tag{3.5}$$

Hence, in view of (3.3), we see that, in response to a control $\varkappa \in K_2$ of player E, the pursuers must point their velocity vectors at a point M on the perpendicular A^*O (Fig.5) such that the phase trajectory will slide along the plane $y = 0$. Denote the length of the segment MO by H.

Obviously, the maximum in (3.4) with respect to $\varkappa$ is reached when $\cos\varkappa \geqslant 0$; therefore, we can determine the point N $(\| NO \| = L)$ on the segment A^*O at which player E "aims". The maximum value of L (denoted by L^*) achieved at controls $\varkappa \in K_2$ is determined from Eq.(3.3) by substituting φ° and ψ°:

$$c\nu L^*/(l_e^2+L^{*2})^{1/2}=b\mu\sin\varphi^\circ+a\lambda\sin\psi^\circ$$

If the parameter L ranges over the interval $[0, L^*]$, then H ranges over the interval $[0, H^\circ]$, where $H^\circ = \|OA^*\|$.

Consider the derivative $d\Gamma/dH$. Taking into account that

$$\sin\varphi = H\,(l_s^2+H^2)^{-1/2},\quad \sin\psi = H\,(l_q^2+H^2)^{-1/2},\quad \sin\varkappa = L\,(l_e^2+L^2)^{-1/2}$$

we obtain

$$d\Gamma/dH = (H-L)\,(b\mu l_s^2\,(l_s^2+H^2)^{-3/2}+a\lambda l_q^2\,(l_q^2+H^2)^{-3/2})$$

The sign of the derivative depends on the sign of $H-L$. The sign changes at points where

$$c\nu H\,(l_e^2+H^2)^{-1/2} = b\mu H\,(l_s^2+H^2)^{-1/2}+a\lambda H\,(l_q^2+H^2)^{-1/2} \tag{3.6}$$

If this equality holds, then $H=L$ and the target points for all players are the same $(M=N)$.

We now investigate the number of roots of Eq.(3.6) in the interval $[0, H^\circ]$.

Since $\nu \leqslant \mu, \lambda$ and $\gamma > 0$, we have $l_s > l_e$ (Fig.4). Dividing by H and noting that $c = a+b$, we obtain

$$a\,(1/\tau_e - 1/\tau_s) + b\,(1/\tau_e - 1/\tau_q) = 0 \tag{3.7}$$

Here τ_e, τ_s, τ_q are the times of motion of the evader and pursuers up to the points $M=N$. Both expressions in parentheses in (3.7) are positive, since $\tau_e < \tau_q$ and $\tau_e < \tau_s$ for any point on A^*O. Therefore, Eq.(3.6) has no roots in $[0, H^\circ]$ (except for the trivial root $H=0$) and the function Γ increases monotonically there (since $H > L$), reaching a maximum at $H = H^\circ$. It can be shown that when $H = H^\circ$ and $L = L^*$ one has a strict inequality $\Gamma < 0$.

This completes the proof of inequality (3.4), and hence also of the fact that the programmed maximin γ is u-stable in the domain D_{21}.

As it is u-stable, the programmed maximin γ is identical with the value ρ^{21} of the differential game (1.1)-(1.4); moreover, in the domain D_{21} interaction of players S and Q is essential in order to attain the optimum result.

The simple behaviour of the value function of the game in phase space is striking. Only the section of the plane $y=0$ belonging to the domain D_{21} possesses singular properties (the scattering plane). At all other points, the value is smooth.

4. We now consider problem (1.1)-(1.4) without fixing the final instant of the game, and determine the minimum time T at which the attainability domains of players S and Q completely cover the attainability domain of E. If this is accomplished by a single player, the game degenerates to a two-person game and the programmed absorption time T is the optimum response time.

If the covering essentially involves both domains, $G_s \cup G_q \supset G_e$, T is the optimum response time in this case too.

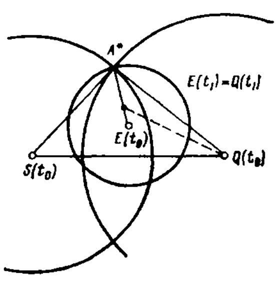

Fig.6

Indeed, suppose that at position $\{t_0, \zeta_0\} \in D_{21}$ (Fig.6) player E selects an extremal target /7/ at the point A^* throughout the time interval $[t_0, T]$. We claim that the pursuers S and Q cannot achieve pointwise capture earlier than at a time $t = T$.

Assume the contrary: at some time $t = t_1 < T$ one of the players (say Q) overtakes player E, moving along the programmed trajectory to the point A^*. Thanks to his superior velocity (1.3), player Q may reach A^* by the time $t = T$. But then

$$\|Q(t_0) - Q(t_1)\| + \|Q(T) - Q(t_1)\| \leqslant \lambda\,(t_1 - t_0) + \lambda\,(T - t_1) =$$
$$\lambda\,(T - t_0) = R_q$$

But since $t_1 < T$ this contradicts the triangle inequality.

The author is indebted to A.G. Pashkov for his formulation of the problem and for his interest.

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THE PROBLEM OF THE STABLE SYNTHESIS OF BOUNDED CONTROLS FOR A CERTAIN CLASS OF NON-STEADY SYSTEMS*

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Developing the results of /1-3/ with regard to the synthesis of bounded controls, a constructive method is given for constructing the controllability function and using the latter to set up a synthesizing control for a certain class of non-steady systems.

1. In this paper the method of Lyapunov functions is employed to solve the following problem of synthesizing bounded controls: given a controlled system

$$x^{\cdot} = f(t, x, u), \quad x \in R^n, \quad u \in \Omega \subset R^r \tag{1.1}$$

it is required to construct a control $u = u(t, x)$ satisfying a given constraint $u \in \Omega$ such that the trajectory $x(t)$ of system (1.1), beginning at an arbitrary point x_0 at time t_0, arrives at the final instant of time $t_0 + T$ $(T = T(t_0, x_0))$ at a preassigned point x_1. The synthesis is said to be stable if x_1 is a rest point (i.e., $f(t, x_1, u_1) = 0$ for some $u_1 \in \Omega$ and any $t \in [t_0, t_0 + T))$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\| x(t) - x_1 \| < \varepsilon$ if $\| x_0 - x_1 \| < \delta$ and $t \in [t_0, t_0 + T)$. Otherewise, the synthesis is said to be unstable. Note that when x_1 is not a rest point the synthesis is, as a rule, unstable.

For example, consider the system $x_1^{\cdot} = x_2 + 1$, $x_2^{\cdot} = u$, $|u| \leqslant 1$. The requirement is that the trajectory reach the origin O $(x_1 = x_2 = 0)$ from an arbitrary point (x_1, x_2). The control solving the synthesis problem is: $u(x) = -1$ if $\varphi \geqslant 0$, $u(x) = 1$ if $\varphi < 0$, where $\varphi = x_1 + (x_2 + 1)\operatorname{sign}(x_2 + 1)/2$. However, any admissible synthesis in this problem is unstable. Indeed, let $x_1(t_0) > 0$. Then a necessary condition for reaching the origin is that at some time t_1, $x_1^{\cdot}(t_1) \leqslant 0$, i.e., $x_2(t_1) \leqslant -1$, whence it follows that any possible synthesis is unstable.

In this paper attention will be confined to the case of stable synthesis. Throughout the sequel it will be assumed, without loss of generality, that $x_1 = 0$. The control synthesis problem will be solved with the help of the controllability function $\Theta(t, x)$ /2/, which plays a role in the stable synthesis problem analogous to that of the Lyapunov function in stability theory.

2. Our solution of the synthesis problem is based on the following theorem.

Theorem 1. Consider the controlled process (1.1). Assume that the vector-function $f(t, x, u)$ is jointly continuous in all variables and, in the domain

$$\{(t, x, u)\colon t_0 \leqslant t \leqslant t_1, 0 < \rho_1 \leqslant \| x \| \leqslant \rho_2, u \in \Omega\}$$

satisfies a Lipschitz condition

$$\| f(t, x'', u'') - f(t, x', u') \| \leqslant L_1(\rho_1, \rho_2)(\| x'' - x' \| + \| u'' - u' \|)$$

Assume that in the closed domain

$$G = J \times \{x\colon \| x \| \leqslant R\} \quad (J = [t_0, t_1]) \tag{2.1}$$

where $0 < R \leqslant +\infty$, there exits a function $\Theta(t, x)$ satisfying the following conditions: